

POSITIVITY OF RELATIVE CANONICAL BUNDLES FOR FAMILIES OF CANONICALLY POLARIZED MANIFOLDS

GEORG SCHUMACHER

ABSTRACT. Given an effectively parameterized family of canonically polarized manifolds the Kähler-Einstein metrics on the fibers induce a hermitian metric on the relative canonical bundle. We use a global elliptic equation to show that this metric is strictly positive. For degenerating families we obtain a singular hermitian metric. Applications concern the curvature of the classical and generalized Weil-Petersson metrics, the hyperbolicity of moduli spaces and the quasi-projectivity of the moduli space of canonically polarized manifolds. We also describe an approach to the case of polarized manifolds with semiample canonical bundle.

1. INTRODUCTION

For any holomorphic family of canonically polarized, complex manifolds, the unique Kähler-Einstein metrics on the fibers define an intrinsic metric on the relative canonical bundle. The construction is functorial in the sense of compatibility with base changes. By definition, its curvature form has at least as many positive eigenvalues as the dimension of the fibers indicates. We show that it is strictly positive, provided the induced deformation is not infinitesimally trivial at the corresponding point of the base.

Actually the first variation of the metric tensor in a family of compact Kähler-Einstein manifolds contains the information about the induced deformation, more precisely, it contains the harmonic representatives of the Kodaira-Spencer classes. The positivity of the hermitian metric will be measured in terms of a certain global function. Essential is an elliptic equation on the fibers, which relates this function to the pointwise norm of the harmonic Kodaira-Spencer forms.

We show that for a degenerating family the positive hermitian metric on the relative canonical bundle extends as a singular positive metric on a certain line bundle. The proof depends on the known C^0 -estimates previously used for the construction of Kähler-Einstein metrics. Furthermore we prove an extension theorem for holomorphic line bundles

with positive, singular hermitian metrics: Such a bundle, given on the complement of a thin analytic set of a reduced complex analytic space, can be extended, if it possesses local extensions as a holomorphic line bundle, and if the curvature form extends as a closed, positive current. This theorem is first applied to the total space of the universal family $\mathcal{X} \rightarrow \mathcal{H}$ over the Hilbert scheme. Next we consider a certain determinant line bundle of the relative canonical bundle, which is defined over the open part of the Hilbert scheme. It carries a Quillen metric, and its curvature form is the generalized Weil-Petersson form. The latter is equal to the fiber integral $\int_{\mathcal{X}/\mathcal{H}} c_1(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)^{n+1}$, where n is the fiber dimension. These quantities descend to the moduli space of canonically polarized manifolds. Again we see that the curvature form extends as a closed, positive current, and the determinant line bundle possesses local extensions to the compactified moduli space (we use the fact that the moduli space is an algebraic space). The extension theorem provides us with a holomorphic line bundle on the compactified moduli space, which is strictly positive and of class C^∞ on the interior, and which extends as a positive, singular hermitian bundle. This implies the quasi-projectivity of the moduli space of canonically polarized manifolds. Acknowledgements.

2. POSITIVITY OF $K_{\mathcal{X}/S}$

Let X be a canonically polarized manifold of dimension n equipped with a Kähler-Einstein metric ω_X . In terms of local holomorphic coordinates (z^1, \dots, z^n) we write

$$\omega_X = \sqrt{-1} g_{\alpha\bar{\beta}}(z) dz^\alpha \wedge d\bar{z}^\beta$$

so that the Kähler-Einstein equation reads

$$(1) \quad \omega_X = -\text{Ric}(\omega_X), \text{ i.e. } \omega_X = \sqrt{-1} \partial\bar{\partial} \log g(z),$$

where $g := \det g_{\alpha\bar{\beta}}$. We consider g as a hermitian metric on the anti-canonical bundle K_X^{-1} .

For any holomorphic family of compact, canonically polarized manifolds $f : \mathcal{X} \rightarrow S$ of dimension n with fibers \mathcal{X}_s for $s \in S$ the Kähler-Einstein forms $\omega_{\mathcal{X}_s}$ depend differentiably on the parameter s . The resulting relative Kähler form will be denoted by

$$\omega_{\mathcal{X}/S} = \sqrt{-1} g_{\alpha\bar{\beta}}(z, s) dz^\alpha \wedge d\bar{z}^\beta.$$

The corresponding hermitian metric on the relative anti-canonical bundle is given by $g = \det g_{\alpha\bar{\beta}}(z, s)$. We consider the real $(1, 1)$ -form

$$\omega_{\mathcal{X}} = \sqrt{-1} \partial\bar{\partial} \log g(z, s)$$

on the total space \mathcal{X} . We will discuss the question, whether $\omega_{\mathcal{X}}$ is a Kähler form on the total space.

The Kähler-Einstein equation (1) implies that

$$\omega_{\mathcal{X}}|_{\mathcal{X}_s} = \omega_{\mathcal{X}_s}$$

for all $s \in S$. In particular $\omega_{\mathcal{X}}$, restricted to any fiber, is positive definite. Our result is the following statement.

Theorem 1. *Let $\mathcal{X} \rightarrow S$ be a holomorphic family of canonically polarized, compact, complex manifolds. Then the hermitian metric on $K_{\mathcal{X}/S}$ induced by the Kähler-Einstein metrics on the fibers is semi-positive and strictly positive in fiber direction. It is strictly positive over points of the base, where the family is not infinitesimally trivial.*

Both the statement of the Theorem and the methods are valid for smooth, proper families of singular (even non-reduced) complex spaces (for the necessary theory cf. [F-S]).

It is sufficient to prove the theorem for one-dimensional families assuming $S \subset \mathbb{C}$. (In order to treat singular base spaces, we can reduce the claim to non-reduced base spaces of embedding dimension one, where the arguments below are still meaningful and can be applied literally.)

We denote the Kodaira-Spencer map for the family $f : \mathcal{X} \rightarrow S$ at a given point $s_0 \in S$ by

$$\rho_{s_0} : T_{s_0} \rightarrow H^1(X, \mathcal{T}_X)$$

where $X = \mathcal{X}_{s_0}$. The family is called *effectively parameterized* at s_0 , if ρ_{s_0} is injective. The Kodaira-Spencer map is induced as edge homomorphism by the short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{X}/S} \rightarrow \mathcal{T}_{\mathcal{X}} \rightarrow f^* \mathcal{T}_S \rightarrow 0.$$

If $v \in T_{s_0} S$ is a tangent vector, say $v = \frac{\partial}{\partial s}|_{s_0}$ and $\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}$ is any lift to \mathcal{X} along X , then

$$\bar{\partial} \left(\frac{\partial}{\partial s} + b^\alpha(z) \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha(z)}{\partial z^\beta} \frac{\partial}{\partial z^\alpha} dz^\beta$$

is a $\bar{\partial}$ -closed form on X , which represents $\rho_{s_0}(\partial/\partial s)$. Observe that b^α is not a tensor on X unless the family is infinitesimally trivial.

We will use the semi-colon notation as well as raising and lowering of indices for covariant derivatives with respect to the *Kähler-Einstein metrics on the fibers*. The s -direction will be indicated by the index s . In this sense the coefficients of $\omega_{\mathcal{X}}$ will be denoted by $g_{s\bar{s}}$, $g_{\alpha\bar{s}}$, $g_{\alpha\bar{\beta}}$ etc.

Next, we define *canonical lifts* of tangent vectors of S as differentiable vector fields on \mathcal{X} along the fibers of f in the sense of Siu [SIU3]. By

definition these satisfy the property that the induced representative of the Kodaira-Spencer class is *harmonic* (cf. also [SCH2]).

Since the form $\omega_{\mathcal{X}}$ is positive, when restricted to fibers, *horizontal lifts* of tangent vectors with respect to the pointwise sesquilinear form $\langle -, - \rangle_{\omega_{\mathcal{X}}}$ are well-defined.

Lemma 1. *The horizontal lift of $\partial/\partial s$ equals*

$$v = \partial_s + a_s^\alpha \partial_\alpha,$$

where

$$a_s^\alpha = -g^{\bar{\beta}\alpha} g_{s\bar{\beta}}.$$

Proposition 1. *The horizontal lift induces the harmonic representative of $\rho_{s_0}(\partial/\partial s)$.*

Proof. The Kodaira-Spencer form of the tangent vector $\partial/\partial s_0$ is given by $\bar{\partial}v|_{\mathcal{X}_s} = a_{s;\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$.

The above equation follows immediately: We consider the tensor

$$A_{s\bar{\beta}}^\alpha := a_{s;\bar{\beta}}^\alpha|_{\mathcal{X}_{s_0}}$$

on X . Then

$$\begin{aligned} g^{\bar{\beta}\gamma} A_{s\bar{\beta};\gamma}^\alpha &= -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} g_{s\bar{\delta};\bar{\beta}\gamma} = -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} g_{s\bar{\beta};\bar{\delta}\gamma} = -g^{\bar{\beta}\gamma} g^{\bar{\delta}\alpha} \left(g_{s\bar{\beta};\gamma\bar{\delta}} - g_{s\bar{\tau}} R_{\bar{\beta}\bar{\delta}\gamma}^{\bar{\tau}} \right) \\ &= -g^{\bar{\delta}\alpha} \left((\partial \log g / \partial s)_{;\bar{\delta}} + g_{s\bar{\tau}} R_{\bar{\delta}}^{\bar{\tau}} \right) = 0. \end{aligned}$$

□

Next, we introduce a *global* function $\varphi(z, s)$, which is the pointwise inner product of the canonical lift v of $\partial/\partial s$ at $s \in S$ with itself with respect to $\omega_{\mathcal{X}}$. Since $\omega_{\mathcal{X}}$ is not known to be positive definite in all directions, $\varphi \geq 0$ is not known at this point.

Lemma 2.

$$\varphi = \langle \partial_s + a_s^\alpha \partial_\alpha, \partial_s + a_s^\alpha \partial_\beta \rangle_{\omega_{\mathcal{X}}} = g_{s\bar{s}} - g_{\alpha\bar{s}} g_{s\bar{\beta}} g^{\bar{\beta}\alpha}$$

Proof. The proof follows from Lemma 1 and

$$\varphi = g_{s\bar{s}} + g_{s\bar{\beta}} a_{\bar{s}}^{\bar{\beta}} + a_s^\alpha g_{\alpha\bar{s}} + a_s^\alpha a_{\bar{s}}^{\bar{\beta}} g_{\alpha\bar{\beta}}.$$

□

Denote by $\omega_{\mathcal{X}}^{n+1}$ the $(n+1)$ -fold exterior product, divided by $(n+1)!$ and by dV the Euclidean volume element in fiber direction. Then the global real function φ satisfies the following property:

Lemma 3.

$$\omega_{\mathcal{X}}^{n+1} = \varphi \cdot g \cdot dV \sqrt{-1} ds \wedge \bar{ds}.$$

Proof. Compute the following $(n+1) \times (n+1)$ -determinant of

$$\begin{pmatrix} g_{s\bar{s}} & g_{s\bar{\beta}} \\ g_{\alpha\bar{s}} & g_{\alpha\bar{\beta}} \end{pmatrix},$$

where $\alpha, \beta = 1, \dots, n$. \square

So far we are looking at *local* computations, which essentially only involve derivatives of certain tensors. The only *global ingredient* is the fact that we are given global solutions of the Kähler-Einstein equation.

The key quantity is the differentiable function φ on \mathcal{X} . Restricted to any fiber it ties together the yet to be proven positivity of the hermitian metric on the relative canonical bundle and the canonical lift of tangent vectors, which is related to the harmonic Kodaira-Spencer forms.

We use the Laplacian operators $\square_{g,s}$ with non-negative eigenvalues on the fibers \mathcal{X}_s so that for a real valued function χ the equation $\square_{g,s}\chi = -g^{\bar{\beta}\alpha}\chi_{;\alpha\bar{\beta}}$ holds.

Proposition 2. *The following elliptic equation holds fiberwise:*

$$(2) \quad (\square_{g,s} + \text{id})\varphi(z, s) = \|A_s(z, s)\|^2,$$

where

$$A_s = A_{s\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} dz^{\bar{\beta}}$$

is the harmonic representative of the Kodaira-Spencer class $\rho_s(\frac{\partial}{\partial s})$ as above.

Proof. The essence to prove an elliptic equation for the tensors that involve derivatives with respect to the parameter space is to eliminate second order such derivatives. This is achieved by the left hand side of (2). First,

$$\begin{aligned} g^{\bar{\delta}\gamma} g_{s\bar{s};\gamma\bar{\delta}} &= g^{\bar{\delta}\gamma} \partial_s \partial_{\bar{s}} g_{\gamma\bar{\delta}} \\ &= \partial_s (g^{\bar{\delta}\gamma} \partial_{\bar{s}} g_{\gamma\bar{\delta}}) - a_s^{\gamma;\bar{\delta}} \partial_{\bar{s}} g_{\gamma\bar{\delta}} \\ &= \partial_s \partial_{\bar{s}} \log g + a_s^{\gamma;\bar{\delta}} a_{\bar{s}\gamma;\bar{\delta}} \\ &= g_{s\bar{s}} + a_{s;\gamma}^\sigma a_{\bar{s}\sigma;\bar{\delta}} g^{\bar{\delta}\gamma}. \end{aligned}$$

Next

$$(a_s^\sigma a_{\bar{s}\sigma})_{;\gamma\bar{\delta}} g^{\bar{\delta}\gamma} = (a_{s;\gamma\bar{\delta}}^\sigma a_{\bar{s}\sigma} + A_{s\bar{\delta}}^\sigma A_{\bar{s}\sigma\gamma} + a_{s;\gamma}^\sigma a_{\bar{s}\sigma;\bar{\delta}} + a_s^\sigma A_{\bar{s}\sigma\gamma;\bar{\delta}}) g^{\bar{\delta}\gamma}.$$

The last term vanishes because of the harmonicity of A_s , and

$$\begin{aligned} a_{s;\gamma\bar{\delta}}^\sigma g^{\bar{\delta}\gamma} &= A_{s\bar{\delta};\gamma}^\sigma g^{\bar{\delta}\gamma} + a_s^\lambda R_{\lambda\gamma\bar{\delta}}^\sigma g^{\bar{\delta}\gamma} \\ &= 0 - a_s^\lambda R_{\lambda}^\sigma \\ &= a_s^\sigma. \end{aligned}$$

□

Proof of Theorem 1. We first show the semi-positivity of the metric.

As $\omega_{\mathcal{X}}$ is positive definite in fiber direction, we need to show only that $\varphi \geq 0$ (or $\varphi > 0$ resp.) For any fixed $s \in S$, let

$$\varphi(z, s) \geq \varphi(z_0, s).$$

Then

$$\varphi(z_0, s) = \|A_s(z_0, s)\|^2 - \square_{g,s}\varphi(z_0, s) \geq \|A_s(z_0, s)\|^2 \geq 0.$$

For any fixed $s \in S$ the function $\varphi|_{\mathcal{X}_s}$ is not identically zero, otherwise by (2) the family had to be infinitesimally trivial at that point.

According to a theorem of Kazdan and De Turck [DT-K], Kähler-Einstein metrics are real analytic (and by the implicit function theorem depend in a real analytic way upon holomorphic parameters). This applies to the function φ .

The above argument shows that the zero set of φ is contained in the set of points, where all components of A_s vanish.

We mention that the integral mean of $\omega_{\mathcal{X}}^{n+1}$ taken over the fibers is equal to the generalized Weil-Petersson form on S (cf. [F-S, Theorem 7.9]).

The strict positivity of φ follows from the proposition below. □

We consider the equation (2) locally. Let $0 \in U \subset \mathbb{C}^n$ be an open subset containing the origin, and $\omega_U = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}}(z) dz^\alpha \wedge d\bar{z}^\beta$ a real analytic Kähler form on U .

Proposition 3. *Let φ and f be real analytic, non-negative, real functions on U . Suppose*

$$(3) \quad \square_{\omega_U} \varphi + \varphi = f$$

holds. If $\varphi(0) = 0$, then both φ and f vanish identically in a neighborhood of 0.

Proof. ¹ It follows from the assumption that φ has a local minimum at the origin, and (3) implies that $\square_{\omega_U} \varphi(0) = 0$ and $f(0) = 0$.

We set $\square = \square_{\omega_U}$ and chose normal coordinates z^α of the second kind for ω_U at 0. Let $\square_0 = -\sum_{\alpha=1}^n \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\alpha}$ be the standard Laplacian so that

$$\square = \square_0 + h^{\bar{\beta}\alpha}(z) \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$$

where the power series expansions of all $h^{\bar{\beta}\alpha}$ have no terms of order zero or one. Also $\square_0 \varphi(0) = 0$. In the following arguments it may be

¹The claim is also a consequence of [P-W, Theorem 6, Chap. 2, Sect. 3].

necessary to replace U by a smaller neighborhood of zero. Then we can say that both $\square\varphi$ and $\square_0\varphi$ are non-positive.

We suppose that φ is not identically zero and let

$$\varphi = \sum_{\ell \geq \ell_0} \varphi_\ell$$

be the homogeneous expansion of φ into polynomials of degree ℓ with $\varphi_{\ell_0} \neq 0$. It follows from the assumption that $\varphi_{\ell_0} \geq 0$.

The homogeneous components of the Laplacians of least possible order are the components of degree $\ell_0 - 2$

$$(\square\varphi)_{\ell_0-2} = (\square_0\varphi)_{\ell_0-2}$$

because of the choice of the coordinates. Suppose that $(\square_0\varphi)_{\ell_0-2} = \square_0(\varphi_{\ell_0})$ vanishes identically. Then the mean value property implies that $\varphi_{\ell_0} \equiv 0$, which contradicts the choice of ℓ_0 .

Now the integral over the sphere $S(r)$ of radius r with respect to the standard (flat) inner product and surface element dA is taken. Let

$$\tilde{\varphi}(r) = \int_{S(r)} \varphi dA.$$

Then

$$0 \geq \int_{S(r)} \square\varphi dA = \int_{S(r)} \square_0\varphi dA + R(r)$$

where the remaining term $R(r)$ is of order at least $\ell_0 + 2n - 1$ in r , whereas the integrals are of order $\ell_0 + 2n - 3$, unless they vanish identically. In the latter case, again the Laplacians are identically zero, implying $\varphi \equiv 0$. We consider the integrated equation (3). The order of $\tilde{\varphi}(r)$ is $\ell_0 + 2n - 1$ so that the lowest order term on the left hand side of

$$\int_{S(r)} \square\varphi dA + \tilde{\varphi}(r) = \int_{S(r)} f dA$$

is $c \cdot r^{\ell_0+2n-3}$, with $c < 0$ contradicting the property of f . \square

3. FIBER INTEGRALS AND QUILLEN METRICS

In this section we refer to the methods how to produce a positive line bundle on the base of a holomorphic family (cf. [F-S]). Let $f : \mathcal{X} \rightarrow S$ be a proper, smooth holomorphic map and $\omega_{\mathcal{X}/S}$ a closed real $(1, 1)$ -form on \mathcal{X} , whose restrictions to the fibers are Kähler forms. Let (\mathcal{E}, h) be a hermitian vector bundle on \mathcal{X} . We denote the determinant line bundle of \mathcal{E} in the derived category by

$$\lambda(\mathcal{E}) = \det f_!(\mathcal{E}).$$

The main result of Bismut, Gillet and Soulé from [BGS] states the existence of a Quillen metric h^Q on the determinant line bundle such that the following equality holds for its Chern form on the base S :

$$(4) \quad c_1(\lambda(\mathcal{E}), h^Q) = \left[\int_{\mathcal{X}/S} td(\mathcal{X}/S, \omega_{\mathcal{X}/S}) ch(\mathcal{E}, h) \right]_2$$

holds. Here ch and td stand for the Chern and Todd character forms.

We will apply the formula in two different situations.

We use the formula for a virtual bundle of degree zero and set $\mathcal{E} = (\mathcal{L} - \mathcal{L}^{-1})^{n+1}$, where (\mathcal{L}, h) is a hermitian line bundle. The term of lowest degree in $ch(\mathcal{E})$ is equal to

$$(5) \quad 2^{n+1} c_1(\mathcal{L})^{n+1}$$

so that the only contribution of the Todd character form in (4) is the constant 1 resulting in the following equality.

$$(6) \quad c_1(\lambda(\mathcal{E}), h^Q) = 2^{n+1} \int_{\mathcal{X}/S} c_1(\mathcal{L}, h)^{n+1}.$$

4. AN EXTENSION THEOREM FOR HERMITIAN LINE BUNDLES

We will need the notion of a singular hermitian metric over a (reduced) complex space. First, we note that by definition an upper semi-continuous function $u : Z \rightarrow [-\infty, \infty)$ on a complex space Z is plurisubharmonic, if its pull-back to any (locally given) analytic curve is subharmonic (or equal to $-\infty$). Let \mathcal{L} be a holomorphic line bundle und Z , then a positive, singular hermitian metric h on \mathcal{L} is defined by the property that the locally defined function $-\log h$ is plurisubharmonic, when pulled back to the normalization of the space. Only in this sense we consider positive $(1, 1)$ -currents on reduced complex spaces.

The following is an existence theorem for a holomorphic line bundle, while there seems to be little control over the actual extension.

For any positive closed $(1, 1)$ -current T on a complex manifold Y the Lelong number at a point x is denoted by $\nu(T, x)$, and for any $c > 0$ we have the associated sets $E_c(T) = \{x; \nu(T, x) \geq c\}$. According to [SIU2, Main Theorem] these are closed analytic sets.

We will apply Siu's decomposition formula from [SIU2] to a given positive, closed current ω . It reads

$$(7) \quad \omega = \sum_{\mu=0}^{\infty} \mu_k[Z_k] + R,$$

where the μ_k are positive numbers, the $[Z_k]$ are currents of integration over irreducible hypersurfaces Z_k , and where R denotes a residual current. The μ_k are the generic Lelong numbers of ω along Z_k . The residual current has the property that $\dim E_c(R) < n - 1$ for every $c > 0$.

Theorem 2. *Let Y be a compact, normal space and $Y' = Y \setminus A$ the complement of a closed analytic, nowhere dense subset. Let L' be a holomorphic line bundle together with a (semi-)positive hermitian metric, which also may be singular. Assume*

- (i) *the curvature current ω' of (L', h') possesses an extension ω to Y as a closed positive current*
- (ii) *the line bundle L' possesses local holomorphic extensions to Y .*

Then there exists a holomorphic line bundle (L, h) with a singular, positive hermitian metric, whose restriction to Y' is isomorphic to (L', h') .

Proof. We first assume that the space Y is smooth. Let $\{U_j\}$ be an open covering of Y with $U'_j = U_j \cap Y'$ such that the line bundle L' is given by a cocycle $g'_{ij} \in \mathcal{O}_Y^*(U'_{ij})$, where $U_{ij} = U_i \cap U_j$ and $U'_{ij} = U_{ij} \cap Y'$. (The local extensions of L' are just trivial line bundles L_i on U_i .) We set $h'_i = h'|_{U'_i}$. Because of the assumptions on the curvature current, we find plurisubharmonic functions ψ_i on U_i such that

$$-\sqrt{-1}\partial\bar{\partial}\log h'_i = \sqrt{-1}\partial\bar{\partial}(2\psi_i|_{U'_i}).$$

Hence

$$h'_i \cdot |e^{f'_i}|^2 = e^{-2\psi_i}|_{U'_i}$$

for some $f'_i \in \mathcal{O}(U'_i)$. We use the $e^{f'_i}$ for a coordinate transformation. Let

$$\tilde{g}'_{ij} = e^{-f'_i} g'_{ij} e^{f'_j}.$$

The metric h' is given on U'_i in the new fiber coordinate by

$$\tilde{h}'_i = h'_i |e^{f'_i}|^2 = e^{-2\psi_i}|_{U'_i}.$$

Now

$$|\tilde{g}'_{ij}|^2 = e^{2\psi_i - 2\psi_j}|_{U'_{ij}}.$$

We will have to change the trivial line bundles L_i in the following sense.

We subtract the (integer) divisorial part in (7) from ω , we write

$$\omega - \sum [\mu_k][Z_k] = \sum (\mu_k - [\mu_k])[Z_k] + R.$$

The divisors Z_k are contained in the set A . Let the $\sum [\mu_k][Z_k]$ be locally given by holomorphic functions $a_i \in \mathcal{O}(U_i)$. We define

$$\hat{g}'_{ij} = \tilde{g}'_{ij} a_i / a_j.$$

We pick a number c with $0 < c < 1$ and consider $\tilde{Y} = Y \setminus E_c(R)$. Restricted to \tilde{Y} , the positive, closed current $\sum(\mu_k - \lfloor \mu_k \rfloor)[Z_k] + R$ has Lelong numbers smaller than one everywhere. Let 2χ be a local $\partial\bar{\partial}$ -potential for this current on \tilde{Y} . The theorem of Bombieri [BO] and Skoda [SK] implies that $e^{-2\chi}$ is locally integrable. Now for the new cocycle \tilde{g}'_{ij} the property

$$|\tilde{g}'_{ij}|^2 \in L^1_{\text{loc}}(U_{ij} \cap \tilde{Y})$$

holds. Hence the \tilde{g}'_{ij} can be extended holomorphically to the sets U_{ij} . As both \tilde{g}'_{ij} and \tilde{g}'_{ji} can be extended, we obtain a cocycle in \mathcal{O}_Y^* , which defines a certain holomorphic line bundle on Y . It follows from the construction that the singular metric extends to the line bundle constructed above with curvature current equal to $\sum(\mu_k - \lfloor \mu_k \rfloor)[Z_k] + R$. On the interior nothing changed. \square

We remark that the theorem also holds for orbifold structures (where a certain finite tensor power of the given line bundles must be taken to ensure that the bundles descend as holomorphic line bundles).

If Y is just a reduced complex space, we still have the following statement.

Proposition 4. *Let Y be a reduced complex space, and $A \subset Y$ a closed analytic subset. Let \mathcal{L} be an invertible sheaf on $Y \setminus A$, which possesses a holomorphic extension to the normalization of Y as an invertible sheaf. Then there exists a reduced complex space Z together with a finite map $Z \rightarrow Y$, which is an isomorphism over $Y \setminus A$ such that \mathcal{L} possesses an extension as an invertible sheaf to Z .*

Proof. Denote by $\nu : \hat{Y} \rightarrow Y$ the normalization of Y , and by $\hat{\mathcal{L}}$ the extension of the pull-back of \mathcal{L} to \hat{Y} . Then, according to [SIU1] the sheaf

$$(\nu_*\mathcal{L})[Y \setminus A] \subset \nu_*\mathcal{L}$$

on Y that is given by the presheaf

$$U \mapsto \{\sigma \in (\nu_*\hat{\mathcal{L}})(U); \sigma|_{U \setminus A} \in \mathcal{L}(U \setminus A)\}$$

is a coherent \mathcal{O}_Y -module consisting of \mathbb{C} -algebras. The analytic spectrum according to Forster [FO] provides the space Z . \square

5. WEIL-PETERSSON METRIC ON HILBERT SCHEMES

AND DOUADY SPACES

In [B-S] and [A-S] a Weil-Petersson metric for Douady spaces and Hilbert schemes was studied. Let (Z, ω_Z) be a Kähler manifold, and

$$(8) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{i} & Z \times S \\ & \searrow f & \downarrow \text{pr}_2 \\ & & S \end{array}$$

a flat holomorphic family of complex submanifolds of Z , parameterized by a complex space S of dimension n . Let s_0 be a distinguished point of S with fiber $X = \mathcal{X}_{s_0}$. The associated Kodaira-Spencer map is

$$\rho_{s_0} : T_{s_0}S \rightarrow H^0(X, \mathcal{N}_{X|Z}).$$

Denote by ω_X the induced Kähler form on X . The Kähler form ω_Z induces a pointwise hermitian inner product $\langle \cdot, \cdot \rangle_{\omega_Z}$ on the normal bundle $\mathcal{N}_{X|Z}$, and by integration a natural inner product on the space of its global holomorphic sections.

Definition 1. *The Weil-Petersson inner product for $v, w \in T_{s_0}S$ equals*

$$\langle v, w \rangle_{PW} = \int_X \langle \rho(v), \rho(w) \rangle_{\omega_Z} \omega_X^n.$$

We recall that the above hermitian inner product is positive definite in directions, where the given family is effective (i.e. for which the Kodaira-Spencer map is not equal to zero), and the corresponding hermitian form on S is induced by the fiber integral

$$(9) \quad \omega_S^{PW} = \int_{\mathcal{X}/S} (\tilde{\omega}^{n+1} | \mathcal{X}),$$

where $n = \dim X$ and $\tilde{\omega}$ is the pull-back of ω_Z to $Z \times S$. The construction is intrinsic, in particular the fiber integral commutes with base change morphisms. The fiber integral construction implies that the Weil-Petersson form ω_S^{PW} is d -closed i.e. *Kähler*.

For Hilbert schemes $S = \mathcal{H}$ and $Z = \mathbb{P}_N$ equipped with the Fubini-Study metric $\omega_Z = \omega^{FS} = c_1(\mathcal{O}_{\mathbb{P}_N}(1), h^{FS})$, the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}_N}(1)$ gives rise to the determinant line bundle in the derived category $\lambda = \det f_!(\mathcal{L} - \mathcal{L}^{-1})^{n+1}$. We invoke the results from Section 3 and get

$$\omega_{\mathcal{H}}^{WP} = \frac{1}{2^{n+1}} c_1(\lambda, h^Q).$$

Next, we consider the component $\overline{\mathcal{H}}$ of the Hilbert scheme corresponding to flat (possibly singular) morphisms.

$$\begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\overline{i}} & \mathbb{P}_N \times \overline{\mathcal{H}} \\ & \searrow \overline{f} & \downarrow pr_2 \\ & & \overline{\mathcal{H}} \end{array}$$

(with flat proper morphism \overline{f}). With $\overline{\mathcal{L}} = \overline{i}^* \mathcal{O}_{\mathbb{P}_N}(1)$ we have an extension $\overline{\lambda} = \det \overline{f}_!((\overline{\mathcal{L}} - \overline{\mathcal{L}}^{-1})^{n+1})$ of λ as a holomorphic line bundle to $\overline{\mathcal{H}}$. We denote the extension of the Weil-Petersson metric given by a fiber integral analogous to (9) by

We define

$$(10) \quad \omega_{\overline{\mathcal{H}}} = \int_{\overline{\mathcal{X}}/\overline{\mathcal{H}}} (\widehat{\omega}^{n+1} | \overline{\mathcal{X}}),$$

where $\widehat{\omega}$ is the pull-back of the Fubini-Study form to $\overline{\mathcal{X}}$.

Generally speaking, the fiber integral is also meaningful for flat embedded families, where only the existence of one non-singular fiber is required. It was shown in [VA1, Lemme 3.4] that $\omega_{\overline{\mathcal{H}}}$ is a d -closed real $(1,1)$ -current (positive in the sense of currents), which possesses a *continuous $\partial\bar{\partial}$ -potential*. So the curvature form of the Quillen metric on λ extends in this sense. From now on we restrict ourselves to the normalization of the Hilbert scheme and of its compactification resp.

Proposition 5. *There exists a holomorphic line bundle $\widetilde{\lambda}$ on $\overline{\mathcal{H}}$ with $\widetilde{\lambda}|_{\mathcal{H}} \simeq \lambda$ and a continuous extension \widetilde{h}^Q of the Quillen metric on λ such that*

$$\omega_{\overline{\mathcal{H}}} = \frac{1}{2^{n+1}} c_1(\widetilde{\lambda}, \widetilde{h}^Q).$$

Proof. We can adopt the proof of Theorem 2. Let $\{U_i\}$ be an open covering of $\overline{\mathcal{H}}$ such that the line bundle $\overline{\lambda}$ is given by a cocycle g_{ij} . Let $U'_i = U_i \cap \mathcal{H}$ and $U'_{ij} = U_{ij} \cap \mathcal{H}$. Let the Quillen metric on $\lambda|_{U'_i}$ be given by h'_i , and let $\sqrt{-1}\partial\bar{\partial} \log h_i = \sqrt{-1}\partial\bar{\partial} \psi_i|_{U'_j}$ for certain *continuous* functions ψ_i on U_i . Hence

$$h'_i = |e^{f'_i}|^2 \cdot (e^{\psi_i}|_{U'_i})$$

for certain holomorphic functions f'_i on U'_i . Now the

$$(|g_{ij}|^2 |e^{f'_i - f'_j}|^2) |U'_{ij} = e^{\psi_i - \psi_j} |U'_{ij}$$

possess continuous extensions to U_{ij} which have no zeroes. This shows that the $g_{ij}e^{f'_i-f'_j}|U'_{ij}$ can be extended as cocycles to $\overline{\mathcal{H}}$ defining $\widehat{\lambda}$ in a way such that the Quillen metrics extend in a continuous way to $\overline{\mathcal{H}}$. \square

Based upon the results of Bismut from [BI] also for certain nodal singularities the Quillen metric of a determinant line bundle can be defined allowing for such singularities. Details are discussed in [A-S].

Next we will use the Weil-Petersson metric on a Hilbert space as a background metric in the relative Kähler-Einstein case.

6. DEGENERATING FAMILIES OF CANONICALLY POLARIZED VARIETIES

In this section we want to show that in a degenerating family the curvature of the relative canonical bundle can be extended as positive closed currents. By definition this is an extension to the normalization. Accordingly, we can restrict ourselves to *normal* base spaces and normalizations of Hilbert spaces. So we can assume that the Hilbert spaces (and compactifications) are constructed in the category of normal complex spaces.

Given a canonically polarized manifold X together with an m -canonical embedding $\Phi = \Phi_{mK_X} : X \hookrightarrow \mathbb{P}_N$, the Fubini-Study metric h^{FS} on the hyperplane section bundle $\mathcal{O}_{\mathbb{P}_N}(1)$ defines a volume form

$$(11) \quad \Omega_X^0 = \left(\sum_{i=0}^N |\Phi_i(z)|^2 \right)^{1/m}$$

on the manifold X , such that

$$\omega_X^0 := -\text{Ric}(\Omega_X^0) = \frac{1}{m} \omega^{FS}|_X,$$

where ω^{FS} denotes the Fubini-Study form on \mathbb{P}_N . According to Yau's theorem, ω_X can be deformed into a Kähler-Einstein metric $\omega_X = \omega_X^0 + \sqrt{-1}\partial\bar{\partial}u$. It solves the equation (1) which is equivalent to

$$(12) \quad (\omega_X^0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega_X^0.$$

We will need the C^0 -estimates for the (uniquely determined) C^∞ -function u .

The deviation of ω_X^0 from being Kähler-Einstein is given by the function

$$(13) \quad F = \log \frac{\Omega_X^0}{(\omega_X^0)^n}.$$

We will use the C^0 -estimate for u from [C-Y, Proposition 4.1] (cf. [AU, K]).

C^0 -estimate. Let \square^0 denote the complex Laplacian on functions with respect to ω_X^0 (with non-negative eigenvalues). Then

$$(14) \quad u + F \leq -\square^0(u).$$

In particular the function u is bounded from above by $\sup(-F)$.

Now we come to the relative situation. In this section we denote the normalization of the compactified Hilbert scheme again by $\overline{\mathcal{H}}$ containing the (normalized) Hilbert scheme by \mathcal{H} . Let

$$(15) \quad \begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\Phi} & \mathbb{P}_N \times \overline{\mathcal{H}} \\ & \searrow f & \downarrow \text{pr}_2 \\ & & \overline{\mathcal{H}} \end{array}$$

Next, we desingularize $\overline{\mathcal{H}}$ in a way that the union of the singular locus and the singular set corresponds to a normal crossings divisor $D \subset \tilde{\mathcal{H}}$. We just take the pull-back $\tilde{\mathcal{X}} = \overline{\mathcal{X}} \times_{\overline{\mathcal{H}}} \tilde{\mathcal{H}}$ of the embedded flat singular family, allowing non-reduced fibers. Let σ be a canonical section of $[D]$, and $|\sigma|$ its locally defined absolute value.

Let $\tilde{\Phi} : \tilde{\mathcal{X}} \rightarrow \mathbb{P}_N \times \tilde{\mathcal{H}}$ be the induced morphism. Let $\mathcal{H}' = \tilde{\mathcal{H}} \setminus D$, and denote by $f' : \tilde{\mathcal{X}}' \rightarrow \mathcal{H}'$ the restricted map. We set $\mathcal{L} = \Phi^* \mathcal{O}_{\mathbb{P}_N \times S}(1)$ and denote by $\tilde{\mathcal{L}}$ its pull-back to $\tilde{\mathcal{X}}$. Furthermore, we assume that $\tilde{\mathcal{L}}|_{\tilde{\mathcal{X}}'} = \mathcal{O}_{\tilde{\mathcal{X}}'}(mK_{\tilde{\mathcal{X}}'/\mathcal{H}'})$.

Next, we denote the relative the initial volume form on $\tilde{\mathcal{X}}'$ by Ω^0 , which again is defined in terms of Φ and the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}_N}(1)$ like in (11). In a similar way $\omega_{\tilde{\mathcal{X}}'}^0$ is defined. We denote by u and F the quantities defined above, depending on the base parameter. Since all quantities on the right hand side are defined in terms of polynomials, it follows immediately that for some positive exponents k and ℓ we have

$$|\sigma(s)|^{2k} \sup \left(\frac{(\omega_{\tilde{\mathcal{X}}'}^0)^n}{\Omega^0} \Big|_{\tilde{\mathcal{X}}'_s} \right) \leq \text{const.}$$

for $s \in \mathcal{H}'$, i.e.

$$|\sigma(s)|^{2k} \sup \{e^{-F(z)}; f(z) = s\} \leq \text{const.}$$

and

$$|\sigma(s)|^{2\ell} \Omega^0 \leq \text{const.}$$

holds. The C^0 -estimate (14) implies that

$$|\sigma(s)|^{2(k+\ell)} e^u \Omega^0 \leq \text{const.}$$

This function is defined locally with respect to the base, i.e. on a set of the form $\tilde{f}^{-1}(U \setminus D)$. Its logarithm is plurisubharmonic over the complement of the divisor D and bounded from above, hence it possesses a plurisubharmonic extension. Now the *current*

$$\sqrt{-1} \partial \bar{\partial} \log(|\sigma(s)|^{2(k+\ell)} e^u \Omega^0)|_{\tilde{f}^{-1}(U \setminus D)} = \sqrt{-1} \partial \bar{\partial} \log(e^u \Omega^0)|_{\tilde{f}^{-1}(U \setminus D)}$$

can be extended to $\tilde{\mathcal{X}}$ as a positive, closed current. The necessary term to be added is basically a multiple of the current of integration over the boundary divisor. The push-forward to \mathcal{X} of the extended current is a positive, closed extension of the original curvature current. We showed the following fact.

Theorem 3. *Let $(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)$ be the relative canonical bundle on the total space over the Hilbert scheme, equipped with the hermitian metric induced by the Kähler-Einstein metrics on the fibers. Then the curvature form extends to the total space over the compact Hilbert scheme $\overline{\mathcal{H}}$ as a positive, closed current $\omega_{\overline{\mathcal{X}}}^{KE}$.*

We observe that stronger estimates are to be expected for slices through boundary points of the Hilbert scheme that correspond to log-canonical singularities.

7. EXTENSION OF THE WEIL-PETERSSON FORM FOR CANONICALLY POLARIZED VARIETIES TO THE COMPACTIFIED HILBERT SCHEME

Following [F-S] for any (smooth) family $f : \mathcal{Z} \rightarrow S$ of canonically polarized manifolds, the generalized Weil-Petersson form ω_S^{WP} on S was proven to be equal up to a numerical factor to a certain fiber integral.

We set $\mathcal{E} = (\mathcal{K}_{\mathcal{Z}/S} - \mathcal{K}_{\mathcal{Z}/S}^{-1})^{n+1}$ in (4). Equation (6) in Section 3 yields

$$(16) \quad c_1(\lambda(\mathcal{E}), h^Q) \simeq \int_{\mathcal{Z}/S} \omega_{\mathcal{Z}}^{n+1},$$

where $\omega_{\mathcal{Z}} = c_1(\mathcal{K}_{\mathcal{Z}/S}, h)$.

On the other hand Lemma 2 together with Proposition 2 imply that the fiber integral equals the generalized Weil-Petersson form:

$$\omega^{WP}(s) = \int_{\mathcal{Z}_s} A_{i\bar{\beta}}^\alpha A_{\bar{\gamma}}^{\bar{\delta}} g_{\alpha\bar{\delta}} g^{\bar{\beta}\gamma} g dV \sqrt{-1} ds^i \wedge ds^{\bar{j}},$$

where the forms

$$A_{i\bar{\beta}}^\alpha \partial_\alpha dz^{\bar{\beta}}$$

are the harmonic representatives of the Kodaira-Spencer classes $\rho_s(\partial/\partial s_i|_s)$, i.e.

$$(17) \quad \omega_S^{WP} \simeq \int_{Z/S} c_1(\mathcal{K}_{Z/S}, h)^{n+1}.$$

Now

$$(18) \quad c_1(\det f_!((\mathcal{K}_{Z/S} - \mathcal{K}_{Z/S}^{-1})^{n+1}), h^Q) \simeq \omega_S^{WP}.$$

We first consider the situation of Hilbert schemes of canonically polarized varieties.

After fixing the Hilbert polynomial and a multiple m of the canonical bundles, which yields very ampleness we consider the universal embedded family over the Hilbert scheme

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i} & \mathbb{P}_N \times \mathcal{H} \\ & \searrow f & \downarrow pr \\ & & \mathcal{H}. \end{array}$$

In this sense we modify the determinant line bundle and consider

$$\lambda = \det f_!((\mathcal{K}_{\mathcal{X}/S}^{\otimes m} - (\mathcal{K}_{\mathcal{X}/S}^{-1})^{\otimes m})^{n+1}).$$

Now we look at the compact Hilbert scheme $\overline{\mathcal{H}}$ including points with singular fibers

$$\begin{array}{ccc} \overline{\mathcal{X}} & \xrightarrow{\bar{i}} & \mathbb{P}_N \times \overline{\mathcal{H}} \\ & \searrow \bar{f} & \downarrow pr \\ & & \overline{\mathcal{H}} \end{array}$$

such that \bar{f} is a flat morphism. Again we let \mathcal{L} be the pull-back of the hyperplane section bundle to $\overline{\mathcal{X}}$. Now the determinant line bundle λ extends to

$$\bar{\lambda} = \det \bar{f}_!((\mathcal{L}^{\otimes m} - (\mathcal{L}^{-1})^{\otimes m})^{n+1})$$

on $\overline{\mathcal{H}}$.

We will need the fact that the construction of the Quillen metric of determinant line bundles can be extended to smooth proper families over base spaces with arbitrary singularities in such a way that $-\log h^Q$ is locally the restriction of a C^∞ function given on a smooth ambient space [F-S, §12]. This C^∞ function is a $\partial\bar{\partial}$ -potential of a real $(1,1)$ -form, which restricts to the generalized Weil-Petersson form.

Next, we want to apply Theorem 3 and consider the fiber integral analogous to (17) for the map $\tilde{f}: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{H}}$.

$$(19) \quad \omega_{\tilde{\mathcal{H}}}^{WP} = \int_{\tilde{\mathcal{X}}/\tilde{\mathcal{H}}} (\omega_{\tilde{\mathcal{X}}}^{KE})^{n+1}.$$

We need to show that the above fiber integral is well-defined.

Theorem 4. *The extended Weil-Petersson form on $\overline{\mathcal{H}}$ given by the fiber integral (19) is a positive, singular $(1, 1)$ -current.*

Proof. Again, the statement is about the pull-back to the normalization, so that we may assume normality.

We use the notation from Section 6 accordingly. We need to consider the fiber integral (19) over the smooth locus \mathcal{H} of the family, i.e. (17), which is strictly positive in effective directions.

We first observe that we can see from the relative version of the Monge-Ampère equation (12), how the Kähler form on the total space, i.e. the curvature form of the relative canonical bundle (induced by the Kähler-Einstein metrics on the fibers) differs from the restriction of the Fubini-Study form on the total space.

$$\begin{aligned} \omega_{\mathcal{X}}^{KE} &:= 2\pi c_1(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h) = \sqrt{-1}\partial\bar{\partial}(\log(\omega_{\mathcal{X}/\mathcal{H}})^n) = \\ &\quad \sqrt{-1}\partial\bar{\partial}(\log e^u \Omega_{\mathcal{X}}^0) = \omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u, \end{aligned}$$

where $\omega_{\mathcal{X}}^0$ stands for the Fubini-Study form pulled back to \mathcal{X} .

We observe that

$$(\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u)^{n+1} = (\omega_{\mathcal{X}}^0)^{n+1} + \sqrt{-1}\partial\bar{\partial} \left(u \sum_{j=0}^n (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u)^{n-j} \right).$$

First, we note that the fiber integral of $(\omega_{\mathcal{X}}^0)^{n+1}$ possesses an extension as a positive, singular hermitian metric (cf. Proposition 5). Next we use the results of Section 6: Near any point of $\overline{\mathcal{H}} \setminus \mathcal{H}$ the potentials u are bounded from above uniformly in the parameter of the base. We consider

$$\begin{aligned} &\int_{\mathcal{X}/\mathcal{H}} \sqrt{-1}\partial\bar{\partial} (u \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u)^{n-j}) = \\ &\quad \sqrt{-1}\partial\bar{\partial} \left(\int_{\mathcal{X}/\mathcal{H}} u \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u)^{n-j} \right) \end{aligned}$$

where now the potential is explicitly given. Consider the integrals

$$\int_{\mathcal{X}_s} u \cdot (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\bar{\partial}u)^{n-j}$$

over the fibers. Now u is uniformly bounded from above, and the integrals

$$\int_{\mathcal{X}_s} (\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1} \partial \bar{\partial} u)^{n-j}$$

are constant in s .

This shows the boundedness of the potential for the singular Weil-Petersson metrics for families of Kähler-Einstein manifolds of negative curvature. \square

8. MODULI OF CANONICALLY POLARIZED MANIFOLDS

In this section we give a short analytic/differential geometric proof of the quasi-projectivity of moduli spaces of canonically polarized manifolds depending upon the variation of the Kähler-Einstein metrics on such manifolds.

We begin with the fact that (after fixing the necessary numerical invariants) we are looking at a component \mathcal{M} of the moduli space of canonically polarized manifolds, endowed with a compactification $\overline{\mathcal{M}}$, which is an algebraic space.

The proof of the quasi-projectivity is an immediate consequence of the previous sections:

On the moduli space \mathcal{M} of canonically polarized manifolds we are given a certain determinant line bundle, equipped with a strictly positive Quillen metric of class C^∞ in the orbifold sense (cf. [F-S]), which is induced by the Kähler-Einstein volume forms. (It was essential to have a functorial construction for holomorphic families for both the line bundle λ and the Quillen metric h^Q in order to descend to the moduli space.)

From here on, it is sufficient to look at (degenerating) families over *normal* complex spaces. In Section 7 we showed that the curvature form of the relative canonical bundle extends as a positive, closed current to the normalization of the total space – the fiber integral that induces the Weil-Petersson form again extends to the normalization of the compactified moduli space.

The Hilbert space approach yielded *local* extensions of the determinant line bundles to a (normal) compactification of the normalized moduli space so that according to Theorem 2 the pull-back of the line bundle to the normalization extends as a holomorphic line bundle with a positive, singular, hermitian metric.

Since we need strict positivity of the hermitian metric over the moduli space itself, we need to change the compactification in order to

ensure that the original line bundle extends – a fact that follows from Proposition 4. Now the criterion [S-T, Theorem 6] is applicable.

Altogether we proved the following fact, which yields the quasi-projectivity of the moduli space [V, V1, V2, KO].

Theorem 5.

- (i) *Let \mathcal{M} be a component of the moduli space of canonically polarized manifolds. Then a tensor power of the determinant line bundle $\det f_!(\mathcal{K}_{X/S}^{\otimes m} - (\mathcal{K}_{X/S}^{\otimes m})^{-1})^{n+1}$ for holomorphic families together with the Quillen metric descends to (λ, h^Q) the moduli space in the orbifold sense.*
- (ii) *The curvature form of the determinant line bundle extends as a (semi-)positive current to a certain compactification $\overline{\mathcal{M}}$.*
- (ii) *A tensor power of the determinant line bundle extends as a line bundle $\widehat{\lambda}$ to a compactification of the moduli space, the Quillen metric gives rise to a singular hermitian metric on $\widehat{\lambda}$, which is strictly positive and of class C^∞ in the orbifold sense over the interior.*

9. QUASI-PROJECTIVE MODULI FOR PROJECTIVE VARIETIES WITH SEMI-AMPLE CANONICAL BUNDLE

In this section we indicate how to apply the methods to projective manifolds with semi-ample canonical bundle.

Definition 2. *A holomorphic line bundle is called semi-ample, if a positive power is generated by global sections.*

The idea is to use moduli spaces of framed manifolds in the sense of [SCH4]. Given a fixed polarized variety, we consider very ample smooth divisors D , which realize the polarization such that for some $m > 0$ the \mathbb{Q} -Cartier divisor $K_X + (m-1)/mD$ is ample. These objects possess a moduli space, with a partial compactification including points, where the underlying manifold X is still smooth but the divisor D may be singular. We consider the induced Galois covering manifolds $X_m \rightarrow X$, which possess Kähler-Einstein metrics of constant negative curvature. These can also be considered as Kähler-Einstein orbifold metrics for the manifolds X . (Although this process does not produce global families, the setup yields both determinant line bundles and Quillen metrics.) The orbifold standpoint provides us with C^0 -estimates, with no finite base change necessary in a way that for the singular hermitian metric on the relative canonical bundles the Lelong numbers are equal to zero – a fact, which persists also, when performing fiber integration. This

provides us with a positive line bundle on the corresponding Hilbert scheme for polarized varieties. So far the process is also functorial, because all objects are intrinsically defined.

The extension of the determinant line bundle as a hermitian line bundle with a singular, positive hermitian metric to the compactified moduli space is done like the proof in the canonically polarized case: For the estimates, we need a background metric, that arises from a Hilbert scheme situation. Here the assumption of the existence of global generators for some (uniform) tensor power of the canonical bundles is crucial.

10. FURTHER APPLICATIONS

We consider the direct image of $\mathcal{K}_{\mathcal{X}/S}^{\otimes 2}$ for families of canonically polarized manifolds.

Theorem 6. *Let S be a complex manifold and $f : \mathcal{X} \rightarrow S$ a holomorphic family of canonically polarized manifolds, equipped with Kähler-Einstein metrics of constant negative curvature. Let the locally free sheaf*

$$f_*(\mathcal{K}_{\mathcal{X}/S}^{\otimes 2})$$

be equipped with the induced L^2 metric. Then

- (i) *the sheaf $f_*(\mathcal{K}_{\mathcal{X}/S}^{\otimes 2})$ is semi-positive in the sense of Nakano, if S is Kähler.*
- (ii) *the sheaf $f_*(\mathcal{K}_{\mathcal{X}/S}^{\otimes 2})$ is positive in the sense of Nakano, if the family is effectively parameterized everywhere.*

The proof is an immediate consequence of our main theorem and a theorem of Berndtsson [B].

10.1. The classical Weil-Petersson metric on Teichmüller space.

It is known from the results of Wolpert that the classical Weil-Petersson metric for families of Riemann surfaces of genus larger than one has negative curvature: According to [WO] the sectional curvature is negative, and the holomorphic sectional curvature is bounded from above by a negative constant. A stronger curvature property, which is related to strong rigidity, was shown in [SCH1]. The strongest result on curvature by Liu, Sun, and Yau now follows immediately from Theorem 1.

Corollary 1 ([Y]). *The Weil-Petersson metric on the Teichmüller space of Riemann surfaces of genus $p > 1$ is dual Nakano negative.*

Proof. Observe that for a universal family $f : \mathcal{X} \rightarrow S$ the classical Weil-Petersson metric on $R^1 f_* \mathcal{T}_{\mathcal{X}/S}$ corresponds to the L^2 metric on its dual bundle $f_*(\mathcal{K}_{\mathcal{X}/S}^{\otimes 2})$, which is Nakano positive according to Theorem 6. \square

10.2. Curvature of the generalized Weil-Petersson metric and related metrics. In this section we present a new approach to questions related hyperbolicity properties of moduli spaces and the existence of non-isotrivial families in the sense of the hyperbolicity conjecture of Shafarevich. We include immediate corollaries to our main theorem which are closely related to known cases (e.g. [B-V, KE-KO, KV1, KV2, M, V-Z1, V-Z2]).

We pick up the notations from Section 1 (in case of a smooth base space S of arbitrary dimension. Let $f : \mathcal{X} \rightarrow S$ be a smooth, proper holomorphic map, whose fibers \mathcal{X}_s , $s \in S$ are canonically polarized varieties of dimension n , equipped with Kähler-Einstein metrics of constant Ricci curvature equal to one. Let

$$\rho_{s_0} : T_{s_0}S \rightarrow H^1(\mathcal{X}_{s_0}, \mathcal{T}_{\mathcal{X}_{s_0}})$$

be the Kodaira-Spencer map for a point $s \in S$. The induced L^2 -metric on the space of infinitesimal deformations is given by integration of the harmonic representatives of the Kodaira-Spencer classes of tangent vectors. These were discussed in Section 2.

Explicitly the Weil-Petersson hermitian inner product is defined as follows: Let (s^1, \dots, s^k) be local holomorphic coordinates on S such that the given base point corresponds to the origin, and let $(z, s) = (z^1, \dots, z^n, s^1, \dots, s^k)$ be local holomorphic coordinates on \mathcal{X} with $f(z, s) = s$.

Let

$$\rho_s \left(\frac{\partial}{\partial s_i} \right) = [A_{i\bar{\beta}}^\alpha \frac{\partial}{\partial z^\alpha} dz^{\bar{\beta}}] \in H^1(\mathcal{X}_{s_0}, \mathcal{T}_{\mathcal{X}_{s_0}})$$

with harmonic representative $A_{i\bar{\beta}}^\alpha$. Then (with the above notations for the Kähler manifold $X = \mathcal{X}_{s_0}$

$$(20) \quad A_{i\bar{\beta};\bar{\delta}}^\alpha = A_{i\bar{\delta};\bar{\beta}}^\alpha$$

$$(21) \quad 0 = g^{\bar{\delta}\gamma} A_{i\bar{\delta};\gamma}^\alpha$$

$$(22) \quad A_{i\bar{\beta}\bar{\delta}} = A_{i\bar{\delta}\bar{\beta}}.$$

The above equation (20) is the $\bar{\partial}$ -closedness, (21) the harmonicity, and (22) reflects the close relationship with the metric tensor.

Theorem 7. *Any compact subspace or relatively compact subset of the moduli space of canonically polarized complex surfaces possesses a complex Finsler orbifold metric, whose holomorphic curvature is bounded by a negative constant.*

In particular, the theorem implies that there exist no non-isotrivial holomorphic families of canonically polarized complex surfaces over the projective line or an elliptic curve.

We will use the fact that the holomorphic curvature of a Finsler metric at a certain point p in the direction of a tangent vector v is the supremum of the curvatures of the pull-back of the given Finsler metric to a holomorphic disk through p and tangent to v (cf. [A-P]). (For a hermitian metric, the holomorphic curvature is known to be equal to the holomorphic sectional curvature).

These facts readily generalize to metrics of orbifold type.

The construction of the Finsler metric is by modifying the generalized Weil-Petersson metric. We recall the formula for its curvature denoting by \square the complex Laplacian on functions and tensors resp. The functions $A_i \cdot A_{\bar{j}}$ are pointwise inner products of Kodaira-Spencer tensors, whereas $A_i \wedge A_{\bar{j}}$ denotes a $(0, 2)$ -form with values in $\Lambda^2 \mathcal{T}_{\mathcal{X}_s}$ (cf. [SCH2]).

Theorem 8 ([SCH2]). *Let $f : \mathcal{X} \rightarrow S$ be a local universal family of canonically polarized manifolds with smooth base space S . Then the curvature tensor of the generalized Weil-Petersson metric equals*

$$\begin{aligned}
 R_{i\bar{j}k\bar{\ell}}^{\text{PW}}(s) = & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{j}}) (A_k \cdot A_{\bar{\ell}}) g \, dV \\
 & - \int_{\mathcal{X}_s} (\square + 1)^{-1} (A_i \cdot A_{\bar{\ell}}) (A_k \cdot A_{\bar{j}}) g \, dV \\
 (23) \quad & - \int_{\mathcal{X}_s} (\square - 1)^{-1} (A_i \wedge A_k) \cdot (A_{\bar{j}} \wedge A_{\bar{\ell}}) g \, dV.
 \end{aligned}$$

For any harmonic Kodaira-Spencer tensor $A = A_{\bar{\beta}}^{\alpha} \frac{\partial}{\partial z^{\alpha}} dz^{\bar{\beta}}$ we denote by $H(A \wedge A)$ the harmonic part of $A \wedge A$.

The theorem implies the following estimate:

Corollary 2. *Let $A = \xi^i A_i$. Then*

$$(24) \quad R_{i\bar{j}k\bar{\ell}}^{\text{PW}}(s) \xi^i \xi^{\bar{j}} \xi^k \xi^{\bar{\ell}} \leq (-2 \|A\|_{WP}^4 + \|H(A \wedge A)\|^2) / \text{vol}(\mathcal{X}_s).$$

Proof. We apply the eigenspace decompositions of the function $A \cdot \bar{A}$ and the tensor $A \wedge A$ with respect to the Laplacians. It was shown in [SCH2] that the eigenspace decomposition of $A \wedge A$ contains no contributions for eigenvalues $\lambda \in (0, 1]$. \square

Now we denote by G the Finsler metric induced by $G_{i\bar{j}}^{WP}$. It is known that the holomorphic curvature of G is equal to the holomorphic sectional curvature of $G_{i\bar{j}}^{WP}$ (cf. [A-P]).

From now on, we assume that the fibers are of *complex dimension two*. The locally free sheaf $R^2 f_* \Lambda^2 \mathcal{T}_{\mathcal{X}/S}$ is dual to $f_* \mathcal{K}_{\mathcal{X}/S}^{\otimes 2}$. The latter, equipped with the induced L^2 -inner product, is Nakano-positive according to Theorem 6 for any effectively parameterized family $f : \mathcal{X} \rightarrow S$. However, at this point, we cannot give any estimate for the curvature because Theorem 1 does not contain any estimates.

We consider the natural morphism

$$(25) \quad \mu : S^2 T_S \rightarrow R^2 f_* \Lambda^2 \mathcal{T}_{\mathcal{X}/S}.$$

In general, we can only say that it induces a Finsler semi-metric on S . If the semi-metric is not identically zero but vanishes only on a thin analytic subset, it is of non-positive holomorphic curvature (considering that the holomorphic curvature of a Finsler metric is defined in terms of holomorphic curves).

We need the following fact. Let C be a holomorphic curve and $G = G(z) dz d\bar{z}$ a hermitian semi-metric, which is positive on the complement of a discrete subset say. Denote by

$$K_G := - \frac{\partial^2 \log G(z)}{\partial z \partial \bar{z}} \Big/ G(z)$$

the (Ricci) curvature. Let $H = H(z) dz d\bar{z}$ be a further such metric.

Lemma 4 (cf. [SCH3, Lemma 3]).

$$(26) \quad K_{G+H} \leq \frac{G^2}{(G+H)^2} K_G + \frac{H^2}{(G+H)^2} K_H.$$

Observe that for $H \equiv 0$ the equation (26) formally still holds.

Let again $f : \mathcal{X} \rightarrow S$ be a local, universal family of canonically polarized manifolds. Then the Weil-Petersson metric determines a Finsler metric G on S , and the dual Nakano negative bundle $R^2 f_* \Lambda^2 \mathcal{T}_{\mathcal{X}/S}$ determines a Hermitian semi-metric $H \not\equiv 0$ for every curve C with $\mu|_C$ not identically zero, since the map μ restricted to C maps $\mathcal{T}_C^{\otimes 2}$ to the hermitian bundle $R^2 f_* \Lambda^2 \mathcal{T}_{\mathcal{X}/S}|_C$ (compatible with base change).

Now, we can use Corollary 2 and Lemma 4 (under the assumption on the base space in Theorem 7) to construct the desired Finsler orbifold metric from a convex sum $G + \gamma H$ of G and H , whose curvature is bounded by a negative constant from above.

The non-existence of non-isotrivial holomorphic families of canonically polarized surfaces over compact curves C of genus zero or one can be seen directly from Theorem 8 and Theorem 1. If the map μ on C is identically zero, then the curvature formula for the Weil-Petersson metric (23) and the estimate (24) imply the claim, if not, Theorem 6 can be applied directly.

Corollary 3. *Let C be a smooth, compact curve and $f : \mathcal{X} \rightarrow C$ a non-isotrivial family of canonically polarized surfaces. Then $g(C) > 1$ or there exists at least one singular fiber.*

Proof. We apply Theorem 8 and Theorem 1: Let $C' \subset C$ be the set of points with regular fibers. Consider the case, where the map μ on $C' \subset C$ is identically zero. Then the curvature formula for the Weil-Petersson metric (23) and the estimate (24) imply the existence of a metric, whose curvature is bounded from above by a negative constant. If $\mu \not\equiv 0$ we apply the Gauß-Bonnet theorem for singular metrics using Theorem 3. Since in this situation, we do not have a negative upper estimate for the curvature, we cannot bound the Lelong numbers at the singularities. So we can only infer the existence of at least one singular fiber, if $g(C) \leq 1$. \square

REFERENCES

- [A-P] Abate, M., Patrizio, G.: Holomorphic curvature of Finsler metrics and complex geodesics. *J. Geom. Anal.* **6**, 341–363 (1996).
- [AU] Aubin, T.: Equation du type de Monge-Ampère sur les variétés Kähleriennes compactes, *C. R. Acad. Sci. Paris* **283**, 119–121 (1976) / *Bull. Sci. Math.* **102**, 63–95 (1978).
- [A] Aust, H.: A criterion for the quasi-projectivity of complex spaces, forthcoming thesis, Marburg.
- [A-S] Axelsson, R., Schumacher, G.: Kähler geometry of Douady spaces. *Manuscr. math.* **121**, 277–291 (2006).
- [B-V] Bedulev E., Viehweg, E.: Shafarevich conjecture for surfaces of general type over function fields. *Invent. math.* **139**, 603–615 (2000).
- [B] Berndtsson, B.: Curvature of vector bundles associated to holomorphic fibrations. *arXiv:math/0511225v2 [math.CV]* 20 Aug 2007.
- [BI] Bismut, J.-M.: Métriques de Quillen et dégénérescence de variétés kähleriennes. *C. R. Acad. Sci. Paris Sér. I. Math.* **319** (1994), 1287–1291.
- [BGS] Bismut, J.-M.; Gillet, H.; Soulé, Ch.: Analytic torsion and holomorphic determinant bundles I, II, III. *Comm. Math. Phys.* **115** (1988), 49–78, 79–126, 301–351.
- [B-S] Biswas, I.; Schumacher, G.: Generalized Petersson-Weil metric on the Douady space of embedded manifolds. *Complex analysis and algebraic geometry*, 109–115, de Gruyter, Berlin, 2000.
- [BO] Bombieri, E.: Algebraic values of meromorphic maps, *Invent. math.* **10** (1970) 267–287 and Addendum *Invent. math.* **11** 163–166 (1970).
- [C-Y] Cheng, S.Y., Yau, S.T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation. *Comm. Pure Appl. Math.* **33**, 507–544 (1980).
- [DT-K] DeTurck, D., Kazdan, J.: Some regularity theorems in Riemannian geometry. *Ann. Sci. Éc. Norm. Supér.* **14** 249–260 (1981).
- [EM] El Mir, H.: Sur le prolongement des courants positifs fermés, *Acta Math.* **153**, 1–45, (1984).

- [FO] Forster, O.: Zur Theorie der Steinschen Algebren und Moduln. *Math. Z.* **97**, 376–405 (1967).
- [F-S] Fujiki, A., Schumacher, G.: The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics. *Publ. Res. Inst. Math. Sci.* **26**, 101–183 (1990).
- [G-R] Grauert, H., Remmert, R.: Plurisubharmonische Funktionen in komplexen Räumen. *Math. Z.* **65** 175–194 (1956).
- [KA] Kawamata, Y.: Characterization of Abelian Varieties. *Compos. Math.* **43** 253–276 (1981).
- [KE-KO] Kebekus, S., Kovács, S.: Families of canonically polarized varieties over surfaces. *Invent. Math.* **172**, 657–682 (2008).
- [K] Kobayashi, R.: Kähler-Einstein metric on an open algebraic manifold. *Osaka J. Math.* **21**, 399–418 (1984).
- [KO] Kollár, J.: Projectivity of complete moduli, *J. of Diff. Geom.* **32** (1990), 235–268.
- [KO-KO] Kollár, J., Kovács, S.: Log canonical singularities are Du Bois. *arXiv:0902.0648*.
- [K-M] Kollár, J., Mori, S.: *Birational Geometry of Algebraic Varieties*. Cambridge University Press 1998. Translated from Souyuuiri Kikagaku published by Iwanami Shoten, Publishers, Tokyo, 1998.
- [KV1] Kovács, S.: Smooth families over rational and elliptic curves. *J. Alg. Geom.* **5** 369–385 (1996).
- [KV2] Kovács, S.: On the minimal number of singular fibres in a family of surfaces of general type. *Journ. Reine Angew. Math.* **487**, 171–177 (1997).
- [M] Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. *J. Alg. Geom.* **4**, 353–361 (1995).
- [M-T] Mourougane, Ch., Takayama, S.: Hodge metrics and the curvature of higher direct images. *arXiv:0707.3551v1 [math.AG]* 24 Jul 2007
- [L-S-Y] Liu, K., Sun, X., Yau, S.T.: Good geometry and moduli spaces, I. To appear.
- [P-W] Protter, M.H.; Weinberger, H.F.: *Maximum-principles in different equations*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., (1967).
- [SCH1] Schumacher, G.: Harmonic maps of the moduli space of compact Riemann surfaces. *Math. Ann.* **275**, 455–466 (1986).
- [SCH2] Schumacher, G.: The curvature of the Petersson-Weil metric on the moduli space of Kähler-Einstein manifolds. Ancona, V. (ed.) et al., *Complex analysis and geometry*. New York: Plenum Press. The University Series in Mathematics. 339–354 (1993).
- [SCH3] Schumacher, G.: Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps. *Math. Ann.* **311**, 631–645 (1998).
- [SCH4] Schumacher, G.: Moduli of framed manifolds, *Invent. math.* **134** 229–249 (1998).
- [S-T] Schumacher, G., Tsuji, H.: Quasi-projectivity of moduli spaces of polarized varieties. *Ann. Math.* **159**, 597–639 (2004).
- [SI] Sibony, N.: Quelques problèmes de prolongement de courants en analyse complexe. *Duke Math. J.* **52**, 157–197 (1985).

- [SIU1] Siu, Y.T.: Absolute gap-sheaves and extensions of coherent analytic sheaves. *Trans. Am. Math. Soc.* **141**, 361–376 (1969).
- [SIU2] Siu, Y.-T. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.* **27**, 53–156 (1974).
- [SIU3] Siu, Y.-T.: Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class. *Contributions to several complex variables*, Hon. W. Stoll, Proc. Conf. Complex Analysis, Notre Dame/Indiana 1984, *Aspects Math.* E9, 261–298 (1986).
- [SIU4] Siu, Y.-T.: Invariance of plurigenera. *Invent. Math.* **134**, 661–673 (1998).
- [SK] Skoda, H.: Sous-ensembles analytiques d'ordre fini ou infini dans \mathbb{C}^n . *Bull. Soc. Math. France* **100**, 353–408 (1972).
- [SO] Sommese, A.: Criteria for quasi-projectivity. *Math. Ann.* **217** 247–256 (1975).
- [VA1] Varouchas, J.: Stabilité de la classe des variétés Kähleriennes par certains morphismes propres. *Invent. math.* **77**, 117–127 (1984).
- [VA2] Varouchas, J.: Kähler spaces and proper open morphisms. *Math. Ann.* **283**, 13–52 (1989).
- [V] Viehweg, E.: Weak positivity and stability of certain Hilbert points I,II,III, *Invent. Math.* **96**, 639–669 (1989). *Invent. Math.* **101** 191–223 (1990), *Invent. Math.* **101** 521–543 (1990).
- [V1] Viehweg, E.: Quasi-projective moduli for polarized manifolds. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* **30**. Berlin: Springer-Verlag, 1995.
- [V2] Viehweg, E.: Compactifications of smooth families and of moduli spaces of polarized manifolds. *arXiv:math/0605093v2*
- [V-Z1] Viehweg, E.; Zuo, K.: On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds. *Duke Math. J.* **118**, 103–150 (2003).
- [V-Z2] Viehweg, E.; Zuo, K.: On the isotriviality of families of projective manifolds over curves, *J. Algebraic Geom.* **10**, 781–799 (2001).
- [WO] Wolpert, S.: Chern forms and the Riemann tensor for the moduli space of curves. *Invent. Math.* **85**, 119–145 (1986).
- [Y] Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, *Commun. Pure Appl. Math.* **31**, 339–411 (1978).

FACHBEREICH MATHEMATIK UND INFORMATIK, PHILIPPS-UNIVERSITÄT MARBURG, LAHNBERGE, HANS-MEERWEIN-STRASSE, D-35032 MARBURG, GERMANY
E-mail address: `schumac@mathematik.uni-marburg.de`